Lade and Kim Elasto-Plastic Constitutive Model: Implementation Into a FEM Code

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SUMMARY: To implement a non-linear stress-strain relationship into a finite element procedure, one can choose between two different ways: returning mapping or time-dependent algorithms. Both algorithms can be used for rate-independent problems. In the latter scheme, an empirical rule is used to determine the time step length. This paper is concerned with the implementation of the elasto-plastic constitutive model proposed by Lade and Kim into a finite element code using both approaches. Emphases will be placed on the time dependent algorithm. Description and implementation will be presented as well as its performance in analyzing the stress-strain behavior of Beliche Dam.

KEY WORDS: Finite Element Method, Constitutive Models.

1 INTRODUCTION

Since Clough and Woodward (1967) the Finite Element Method (FEM) has been widely used in geotechnics to analyze foundations, dams, excavations and tunnels, solving problems such as stress-strain, seepage and settlement.

One big shortcoming in this analysis soon was pointed out as being the material constitutive stress-strain relationship. In this work, the authors have selected the constitutive model proposed by Lade and Kim (1988a, b, c) for its relative simplicity and wide acceptance in the academic field.

Two main approaches can be used to implement a constitutive relationship in a FEM code: returning mapping and time dependent algorithms. Each of them present different advantages and disadvantages, e.g.: in the returning map algorithm, there is no need to of dealing with fictitious time steps and in the time dependent scheme, there is no need to calculate the gradient of the yield function.

2 LADE & KIM CONSTITUTIVE MODEL

Lade and Duncan (1975) presented an elasto-plastic model to represent the behavior of non-cohesive soils. The formulation was based on experimental observations from cubical triaxial tests performed in sand by Lade (1972) and Lade and Duncan (1973). This model considers only one conical yield surface.

In 1977, Lade developed another model able to represent the plastic deformations that take place during proportional loading $\left(\frac{\sigma_1}{\sigma_3}=cte\right)$. This calculation was possible with the introduction of a second yield surface: a cap surface.

Afterwards, verifying that in terms of effective stress, the behavior of normally consolidated clays observed in laboratory tests are quite similar, Lade (1979) extended this formulation for clays.

More recently, a new model was proposed by Lade and Kim (Lade and Kim, 1988a and 1988b and Kim and Lade 1988) which utilizes of a unique yield surface incorporating both conical and cap surfaces. The model employs the same failure criteria introduced in 1977, but its formulation has some advantages over the previous one. For example the more recent formulation more closely reproduces the behavior of overconsolidated soils and is easier to implement in a finite-element code.

2.1 Elastic deformation

The elastic strain component is calculated using Hooke's law. Among the many ways to calculate the Young modulus, the one selected here was proposed by Janbu (1963), which relies on an unloading-reloading modulus:

$$E_{ur} = K_{ur} p_a \left(\frac{\sigma_3}{p_a} \right)^n \qquad (1)$$

where K_{ur} and n are parameters determined from unloading-reloading diagrams, p_a is the atmospheric pressure and σ_3 is the actual confining pressure. The Poisson's ratio is kept constant in this approach.

2.2 Failure criteria

In the model presented by Lade (1977) a curved failure surface was developed which is expressed in terms of the invariants of the Cauchy stress tensor as:

$$f_n = \left(\frac{I_1^3}{I_3} - 27 \right) \left(\frac{I_1}{p_a} \right)^n \qquad and \qquad (2)$$

$$f_n = \eta_1 \qquad at\ \ failure.$$

where the first and third invariant are:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 = \sigma_x + \sigma_y + \sigma_z \qquad (3)$$

and

$$I_3 = \sigma_1 \sigma_2 \sigma_3 = \sigma_x \sigma_y \sigma_z + \tau_{xy} \tau_{yz} \tau_{zx} + \tau_{yx} \tau_{zy} \tau_{xz} - \left(\sigma_x \tau_{yz} \tau_{zy} + \sigma_y \tau_{zx} \tau_{xz} + \sigma_z \tau_{xy} \tau_{yz} \right) \qquad (4)$$

and m and η_1 are model specific parameters.

2.3 Plastic Potential Function

Saint Venant (1870) proposed that axes of stresses coincide with the principal axes of incremental strains (but not with those of total plastic strain). There can also be stated a coincidence in terms of incremental plastic strains because the principal axes of incremental elastic strain coincide with those of incremental stresses. Based on this observation, Melan (1938) suggested using a plastic potential function for determination of the incremental plastic strain components

$$\left\{ d\varepsilon^p \right\} = d\lambda \left\{ \frac{\partial g}{\partial \sigma} \right\} \qquad (5)$$

Based on experimental data, Kim and Lade (1988) proposed the following plastic potential function:

$$g_p = \left[\psi_1 \frac{I_1^3}{I_3} - \frac{I_1^2}{I_2} + \psi_2 \right] \left[\frac{I_1}{p_a} \right]^{\mu} \qquad (6)$$

where I_2 is the second invariant given by:

$$I_2 = -\left(\sigma_1 \sigma_2 + \sigma_1 \sigma_3 + \sigma_2 \sigma_3 \right) = \tau_{xy} \tau_{yz} + \tau_{yz} \tau_{zy} + \tau_{zx} \tau_{xz} - \left(\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x \right) \qquad (7)$$

and ψ_1, ψ_2 and μ are parameters, such that: ψ_1 is a weighting factor between the triangular and circular shape in the octahedral plane; ψ_2 determines where this curve will intercept the hydrostatic axis and μ controls the curvature of meridians.

Figure 1 shows the plastic potential contours for three different values of g_p for fine silica sand using the parameters proposed by Lade and Kim (1988b).

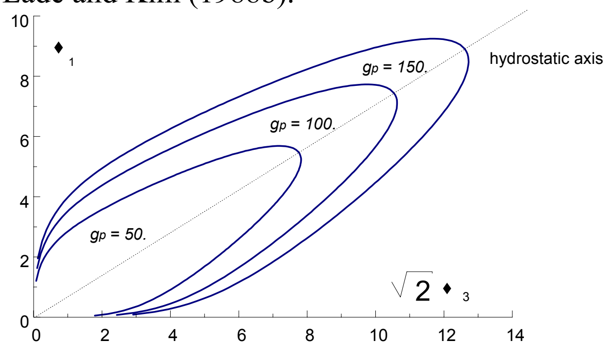

Figure1.1 - Plastic potential contours for different values of g_p - fine silica sand

Taking equation 6 into the flow equation (5), one gets for the plastic strains:

$$\begin{Bmatrix} d\varepsilon_x^p \\ d\varepsilon_y^p \\ d\varepsilon_z^p \\ d\gamma_{yz}^p \\ d\gamma_{zx}^p \\ d\gamma_{xy}^p \end{Bmatrix} = d\lambda_p \left[\frac{I_1}{p_a}\right]^\mu \cdot \begin{Bmatrix} G(\sigma_y+\sigma_z)\frac{I_1^2}{I_2^2} - \psi_1(\sigma_y\sigma_z - \tau_{yz}^2)\frac{I_1^3}{I_3^2} \\ G(\sigma_z+\sigma_x)\frac{I_1^2}{I_2^2} - \psi_1(\sigma_z\sigma_x - \tau_{zx}^2)\frac{I_1^3}{I_3^2} \\ G(\sigma_x+\sigma_y)\frac{I_1^2}{I_2^2} - \psi_1(\sigma_x\sigma_y - \tau_{xy}^2)\frac{I_1^3}{I_3^2} \\ 2\frac{I_1^2}{I_2^2}\tau_{yz} - 2\psi_1(\tau_{xy}\tau_{zx} - \sigma_x\tau_{yz})\frac{I_1^3}{I_3^2} \\ 2\frac{I_1^2}{I_2^2}\tau_{zx} - 2\psi_1(\tau_x\tau_{yz} - \sigma_y\tau_{zx})\frac{I_1^3}{I_3^2} \\ 2\frac{I_1^2}{I_2^2}\tau_{xy} - 2\psi_1(\tau_{yz}\tau_{zx} - \sigma_z\tau_{xy})\frac{I_1^3}{I_3^2} \end{Bmatrix} \tag{8}$$

where

$$G = \psi_1(\mu+3)\frac{I_1^2}{I_3} - (\mu+2)\frac{I_1}{I_2} + \psi_2\,\mu\frac{1}{I_1} \tag{9}$$

Considering that only the principal stresses occurs:

$$\begin{Bmatrix} d\varepsilon_1^p \\ d\varepsilon_2^p \\ d\varepsilon_3^p \end{Bmatrix} = d\lambda_p \left[\frac{I_1}{p_a}\right]^\mu \begin{Bmatrix} G - \frac{I_1^2}{I_2^2}(\sigma_2-\sigma_3) - \psi_1\frac{I_1^3}{I_3^2}\sigma_2\sigma_3 \\ G - \frac{I_1^2}{I_2^2}(\sigma_3-\sigma_1) - \psi_1\frac{I_1^3}{I_3^2}\sigma_3\sigma_1 \\ G - \frac{I_1^2}{I_2^2}(\sigma_1-\sigma_2) - \psi_1\frac{I_1^3}{I_3^2}\sigma_1\sigma_2 \end{Bmatrix} \tag{10}$$

One important aspect in plasticity formulation is the irreversibility condition proposed by Prager (1949). In order to fulfill this condition the plastic work must be positive every time that there is a change in the plastic strains:

$$dW_p = \sigma_{ij}d\varepsilon_{ij}^p = \sigma_{ij}d\lambda_p\frac{\partial g_p}{\partial\sigma_{ij}} \geq 0 \tag{11}$$

For this to happen it is necessary that the plastic potential surface is convex with regard the principal stress space origin. This can be achieved if the right parameters are used. Considering that the plastic potential function is a homogeneous function of the μ^{th} degree (Wylie and Barrett, 1982),

$$\frac{\partial g_p}{\partial\sigma_{ij}}\sigma_{ij} = \mu g_p$$

equation 10 can be written as:

$$dW_p = \mu g_p d\lambda_p \tag{12}$$

Since $d\lambda_p > 0$ the irreversibility condition requires:

$$\mu g_p \geq 0 \tag{13}$$

If g_p is negative, the plastic strain increment will be directed inside the plastic potential surface, an impossible condition. So a negative value of g_p is not appropriate,. and the only solution satisfying the previous inequality is:

$$\mu > 0 \quad and \quad g_p \geq 0 \tag{14}$$

But $\frac{I_1^3}{I_3} \geq 27$, $\left(-\frac{I_1^2}{I_2}\right) \geq 3$ and $\frac{I_1}{p_a} > 0$, so the irreversibility condition can be presented as the following conditions:

$$\mu > 0 \quad and \quad \psi_2 > -(27\psi_1+3) \tag{15}$$

2.4 Yield function

The identification of yield points from experimental tests and the combination of these points to define a yield surface is not an easy task. Because, plasticity is a continuous process in frictional materials there is no yield point in primary loading. Further difficulty arise from the fact that yield surfaces must be associated with hardening/softening parameters which will define the magnitude of the plastic strain increments.

Lade and Kim (1988a) proposed a work-hardening formulation in which each yield surface is an "iso-plastic-work" curve. From many experimental observations, they proposed the following formulation to find the plastic work generated by changing the isotropic pressure:

$$W_p = c p_a \left(\frac{I_1}{p_a} \right)^p \tag{16}$$

where c and p are parameters, p_a is the atmospheric pressure and I_1 is the first stress invariant.

The subsequent equations describe curves having the same plastic work, or in this case, yield surfaces:

$$F(\sigma_{ij}, W_p) = f_p'(\sigma_{ij}) - f_p''(W_p) \tag{17}$$

where,

$$f_p' = \left(\psi_1 \frac{I_1^3}{I_3} - \frac{I_1^2}{I_2} \right) \left(\frac{I_1}{p_a} \right)^h e^q \tag{18}$$

in which ψ_1, h and q are model parameters and:

$$f_p'' = \left(\frac{1}{D} \right)^{\frac{1}{\rho}} \left(\frac{W_p}{p_a} \right)^{\frac{1}{\rho}} \tag{19}$$

where D and ρ will be defined later.

The shape of the proposed yield surface can be seen in figure 2.

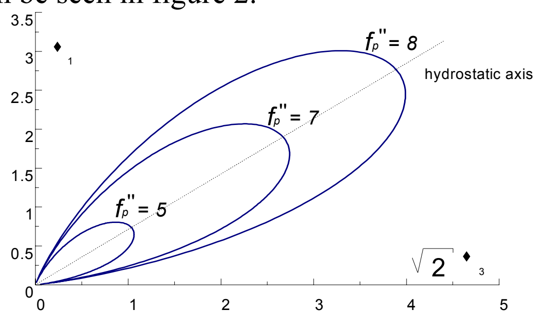

Figure 2 - Yield surfaces for different values of f''$_p$ for fine silica sand

The parameter q is dependent of the stress state, such that:

q = 0 during isotropic compression

0 < q < 1 during hardening

q = 1at failure

For isotropic compression ($q = 0$) equation 1.21 can be written as:

$$f_p' = (27\psi_1 + 3) \left(\frac{I_1}{p_a} \right)^h \tag{20}$$

Making this equation equal to equation 19 (in other words, writing the yielding function for isotropic loading) and substituting in this new equation the value of the plastic work given by 16:

$$D = \frac{c}{(27\psi_1 + 3)^\rho} \tag{21}$$

and

$$\rho = \frac{p}{h} \tag{22}$$

2.5 Parameter determination

The parameters used in this analyzes were obtained by a evolutionary algorithm as explained in de Melo and Azevedo (1995). Table 1 shows the parameters used here in.

	Kur	n	m	η_1	c	p	ψ_2	μ	h	α
sound rockfill	2000	0.10	0.81	1154	0.000092	1.99	-2.75	3.74	1.12	0.33
rockfill	10000	0.10	0.64	385	0.000067	2.44	-2.96	3.11	1.63	0.40
filter	10000	0.10	0.40	146	0.00245	1.27	-2.91	2.76	1.58	0.16
core	256	0.64	0.85	150	0.00048	1.69	-3.02	2.09	0.51	1.19
foundation	2000	0.10	0.25	61	0.00094	1.68	-2.88	2.29	1.08	0.35

Table 1 - Parameters used in the analysis

3 FEM implementation

3.1 Return mapping algorithm

There are two main algorithms that have been proposed using this technique: the closest point projection and the cutting plane algorithm. Due to its relativity simplicity the latter scheme is used. For the closest point algorithm refer to Hughes (1987). For the cutting plane algorithm refer to Simo and Ortiz(1985).

To assemble the stiffness matrix, one can use:

$$\left[E_{ep}\right] = [D] - [D]\left\{\frac{\partial g_p}{\partial \sigma}\right\}\left\{C_\lambda\right\}^{\mathbf{T}} \qquad (23)$$

where,

$$\left\{C_\lambda\right\}^{\mathbf{T}} = \frac{\left\{\frac{\partial F}{\partial \sigma}\right\}^{\mathbf{T}}[D]}{\left\{\frac{\partial F}{\partial \sigma}\right\}^{\mathbf{T}}[D]\left\{\frac{\partial g_p}{\partial \sigma}\right\} - \frac{\partial F}{\partial W_p}\{\sigma\}^{\mathbf{T}}\left\{\frac{\partial g_p}{\partial \sigma}\right\}} \qquad (24)$$

As can be seen, the derivatives of both plastic potential and yield function are needed. The full assembly of this matrix is presented in de Melo (1995).

3.2 Time dependent

Perzyna (1963, 1966) has proposed a theory which is able to describe simultaneously, two material behaviors: plasticity and rheology. Viscous strains are solely time dependent. For this reason they have no distinct yield surface, since they occur also in the elastic region of the stress space. Plastic strains will only occur when the stress state is beyond the yield surface ($F > 0$). Those strains need time to develop. Therefore stress states $F > 0$ are valid for time $t \neq \infty$.

In the model of Lade & Kim viscous strains are not taken into account, so the theory can be simplified to

$$\dot{\varepsilon}_p = 1\frac{1}{\eta}\langle f\rangle\frac{\partial g}{\partial \sigma_{ij}} \qquad (25)$$

where, $\dot{\varepsilon}_p$ is the plastic strain increment, $\langle f\rangle$ is the Heaviside-function { $\langle f\rangle = 0$ for $f < 0$; $\langle f\rangle = F$ for $f > 0$ } and g is the plastic potential function given in equation 6.

The viscosity parameter η is not known in this constitutive model, so η can be set to 1.0. It must be noted that the time then becomes fictitious parameter necessary to obtain a convergent solution.

The visco-plastic strain increment during time step Δt is

$$^{\Delta t}\boldsymbol{\varepsilon}_p = \dot{\boldsymbol{\varepsilon}} \cdot \Delta t \qquad (26)$$

and the corresponding visco-plastic stress increment for this time step is

$$^{\Delta t}\boldsymbol{\sigma}_p = \mathbf{D} \cdot {}^{\Delta t}\boldsymbol{\varepsilon}_p \qquad (27)$$

in which D is the elasticity matrix.

So the total stress for this time step is

$$^{t+\Delta t}\sigma = {}^{t+\Delta t,pg}\sigma - {}^{\Delta t}\sigma_p \qquad (28)$$

where $^{t+\Delta t}\sigma$ is the stress at time $t + \Delta t$ and $^{t+\Delta t,pg}\sigma$ is the stress at time $t + \Delta t$ proposed from global stiffness matrix. The visco-plastic stress increment $^{\Delta\tau}\sigma_p$ does not occur in nature. It is only a magnitude needed for the calculation of the residual forces $\{\Delta R\}$.

3.2.1 Time step length

As mentioned above, the magnitude of the time step is used to control the calculation. At time steps above a certain threshold, oscillating results will be obtained indicating instability of the process. For some models, critical time step duration have been found, but not for the general case.

Zienkiewicz and Cormeau (1974) proposed an empirical rule for a variable time step duration where the anticipated strains are related to the total accumulated strains.The second strain invariant is convenient to use as a scalar for the magnitude of the strain

$$\bar{\varepsilon} = \sqrt{\left[\frac{2}{3}\left(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2\right) + \frac{1}{3}\left(\varepsilon_{xy}^2 + \varepsilon_{xz}^2 + \varepsilon_{yz}^2\right)\right]} \tag{29}$$

in which, $\bar{\varepsilon}$ is the second strain invariant and ε_{ij} are the components of the strain tensor.

The proposed time step for the next iteration step is now determined by the relation

$$^{\Delta t}\bar{\varepsilon}^{vp} \cdot \Delta t = \bar{\varepsilon}^{tot} \cdot \tau \tag{30}$$

where $^{\Delta t}\bar{\varepsilon}^{vp}$ is the visco-plastic strain increment for the current time step (expressed as a scalar), Δt is the proposed time step for the next iteration step, $\bar{\varepsilon}^{tot}$ is the total accumulated strain (a scalar) and τ is a scalar, around $0.10 \sim 0.15$ for low stress level; $0.01 \sim 0.05$ near collapse.

The new time step is the minimum of Δt over all Gauss points and it is further limited by the following relation to the current time step duration:

$$\Delta t^{n+1} \leq 1.5 \cdot \Delta t^{n} \tag{31}$$

Although this proposal for the time step may work well for open yield surfaces along the hydrostatic axis (e.g. Drucker-Prager and Mohr-Coulomb, for which it was developed) it does not fit for yield surfaces, which are closed along the hydrostatic axis.

Furthermore, it seems paradoxical that the proposed time step duration is independent of the number of load steps. Usually, load steps are increased when a more accurate calculation is required. These two problems can be avoided by using only the accumulated strains within one load step instead of the total accumulated strains. (Hartl, 1997)

3.3 Elastic Iteration

The elastic behavior of soil is not linear. As a result, every element (or more precisely, every Gauss point).changes Young's modulus with every change of stress state. Updating the global stiffness matrix after each load step with the new Young's moduli can capture this. But resolving the global stiffness matrix is computationally expensive. Thus, it is reasonable to correct the stress-strain relation due to changes in Young's modulus by using the same framework as for plastic iterations.

It is know from visco-plastic theory that at $t = 0$ purely elastic behavior is assumed over the entire domain. Therefore in this program iteration for the change in elastic behavior is done first, and then the visco-plastic iteration is begun.

$$\Delta\sigma = (\sigma_1 - \sigma_{ec})\cdot f_t \tag{32}$$

with $\Delta\sigma$ being the stress increment whose Young's modulus will be corrected within the present iteration step; σ_1 the proposed stress vector obtained from global stiffness matrix; σ_{ec} the stress vector prior to the correction of Young's modulus and f_t the arbitrary factor $0 < f_t \leq 1$, to get a convergent solution

The stress state is stored away until the Young's modulus is corrected within this step

$$\sigma_{ec} = \sigma_{ec} + \Delta\sigma \tag{33}$$

and the difference in the strain-increment is determined by

$$\varepsilon_{\Delta YM} = (C_{cYM} - C_{gYM})\cdot\Delta\sigma \tag{34}$$

with $\varepsilon_{\Delta YM}$ being the divergent strain increment due to change of Young's modulus (comparable to the plastic strain increment); C_{cYM} the compliance matrix with Young's modulus from the current stress state and C_{gYM} the compliance matrix with Young's modulus from the global stiffness matrix

$$\Delta\sigma_{res} = D_{gYM}\cdot\varepsilon_{\Delta YM} \tag{35}$$

where $\Delta\sigma_{res}$ is the residual stresses and D_{gYM} the elasticity matrix with Young's modulus from the global stiffness matrix.

Finally, the residual stresses are mapped to the element nodes

$$\{R\} = [B] \cdot \{\Delta\sigma_{res}\} \qquad (36)$$

in which B is the strain displacement matrix.

This iteration is performed over all Gauss points within one load step. The exit condition requires that σ_{ec} and σ_1 are close enough (e.g. within 1‰) for every Gauss point.

4 RESULTS

Figure 3 shows the FEM mesh (116 eight nodes isoparametric elements)used here as well as the material distribution within Beliche Dam. The material characteristics and a better description of this dam can be found in Veiga Pinto (1982, 1983) and Naylor et al. (1986). Table 2 shows the unit weights used in this analysis.

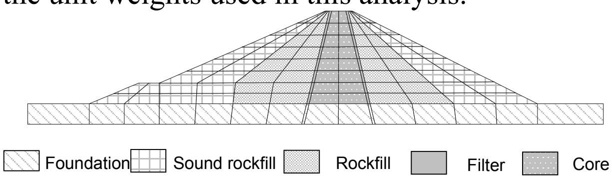

Figure 3 - FEM mesh and material distribution

The turn-on-gravity method was used, as it is considered more appropriate for backfills and dams.

	sound rockfill	rockfill	filter	core	foundation
γ (kN/m3)	20.5	18.2	21.7	21.8	17.9

Table2-Unit weights for Beliche dam's materials

As can be seen on figures 4, 5 and 6 the result is very consistent with those found by de Melo and Azevedo (1995). This strong agreement justifies the described procedure as being valid.

Figure 4 - Vertical stress distribution (Kpa)

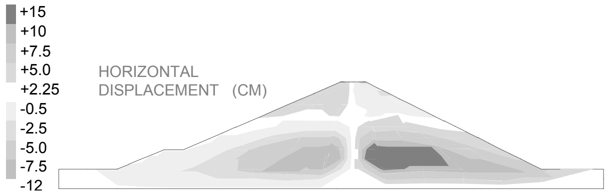

Figure 5 - Horizontal displacement (cm)

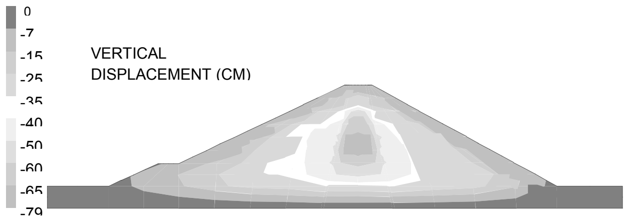

Figure 6 - Vertical displacement (cm)

5 CONCLUSIONS

Choosing an algorithm to use in a FEM code is not an easy task. It depends not only on the constitutive model but also on what kind of problem one wants to solve. In this work a geotechnical case (Beliche Dam) was analyzed using a time dependent (visco-plastic) approach. The results were very consistent with those obtained from a return mapping algorithm (de Melo and Azevedo, 1995). Even though further examinations have to be made in order to analyze other loading conditions, the results presented herein are very encouraging.

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